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1980 J. Phys. A: Math. Gen. 13 2903

(http://iopscience.iop.org/0305-4470/13/9/017)

J. Phys. A: Math. Gen. **13** (1980) 2903–2911. Printed in Great Britain

Dimensional analysis of subtracted-out Feynman integrands

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Received 24 October 1979

Abstract. A rigorous dimensional analysis of Bogoliubov subtracted-out type Feynman integrands is carried out, giving sufficient conditions for determining the asymptotic behaviour of the corresponding integrals in the ultraviolet region in Euclidean space.

1. Introduction

The purpose of this work is to carry out a dimensional analysis of Bogoliubov type subtractions (Bogoliubov and Parasiuk 1957, Parasiuk 1960, Hepp 1966) of Feynman integrands (Zimmermann 1969) R. We give sufficient conditions for determining the asymptotic polynomial *and* logarithmic behaviour of renormalised amplitudes $A = \int R$ in the ultraviolet region in Euclidean space by the application of the power-counting theorem in the Weinberg–Fink sense (Weinberg 1960, Fink 1968) and by determining the class of the so-called maximising subspaces (Manoukian 1978) for the bound of A relative to certain parameters in the theory becoming large. The key result is that we obtain precise conditions under which an integrand R, with its complicated structure of subtractions, may take on its *maximum* dimensionality (§ 2), in conformity with the power-counting theorem, when all or some of the external momenta of a graph become large and, in general, at different rates in Euclidean space.

2. Dimensional analysis

Let $4n$ and $4n'$ denote the number of integration variables and the number of the independent external momentum components, respectively, associated with the renormalised integrand R corresponding to a proper and connected graph G. We combine the $4(n+n')$ variables as the components of a $4(n+n')$-vector in a Euclidean space $R^{4(n+n')}$ in the standard manner (Weinberg 1960). Let $\boldsymbol{P} \in R^{4(n+n')}$ be a vector such that each of the integration and external variables may be written as a linear combination of the components of $\boldsymbol{P}$.

A line l in G, carrying a momentum Q_l, will be represented in the form of a polynomial in Q_l times $[Q_l^2 + \mu_l^2 - \mathrm{i}\epsilon(\boldsymbol{Q}_l^2 + \mu_l^2)]^{-1}$. As we are working in Euclidean space we drop the iϵ factor throughout. We assume as a sufficiency condition that $\mu_l^2 > 0$ for each line l in G, where μ_l denotes the mass carried in the line l. The momentum Q_l may be written as $Q_l = k_l + q_l$, where q_l is a function of the external momenta of G only, and k_l is a function of the integration variables relative to G only.

For each line l in G and for each component Q_l of the corresponding 4-vector, we introduce a vector $\boldsymbol{V}_l$ in $R^{4(n+n')}$ such that $\boldsymbol{V}_l \cdot \boldsymbol{P} = Q_l$. Similarly, we introduce $\boldsymbol{V}$ vectors corresponding to q_l and k_l. To simplify the notation we denote by $\boldsymbol{V}_l^{(1)}$ and $\boldsymbol{V}_l^{(2)}$ the $\boldsymbol{V}$ corresponding to k_l and q_l respectively, relative to the graph. In other words, and quite generally, in reference to a given subdiagram $g \subseteq G$, we will refer to the $\boldsymbol{V}_l^{(1)}$ and $\boldsymbol{V}_l^{(2)}$ associated with its line as the $\boldsymbol{V}$ associated with the *internal* and *external* variables of g respectively. Let I be an arbitrarily chosen and fixed $4n$-dimensional subspace of $R^{4(n+n')}$ associated with the integration variables, and E a subspace of $R^{4(n+n')}$, disjoint from I such that $R^{4(n+n')} = I + E$, with $\Lambda(I)$ the projection operation along I on E. (For more details on definitions and notation see Weinberg 1960.)

Let S_r be a subspace of E spanned by an arbitrarily chosen set of independent vectors $\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots, \boldsymbol{L}_r$: $S_r = \{\boldsymbol{L}_1, \ldots, \boldsymbol{L}_r\}$. We may then write the dependence of the amplitude $A(p_1^0, \ldots, p_1^3, \ldots, p_{n'}^3)$ on its momentum components, as the components of a vector in E, in the general form

$$A(\boldsymbol{L}_1\eta_1\eta_2 \ldots \eta_m + \boldsymbol{L}_2\eta_2 \ldots \eta_m + \ldots + \boldsymbol{L}_r\eta_r \ldots \eta_m + \ldots + \boldsymbol{L}_m\eta_m + \boldsymbol{C}) \tag{1}$$

where $S_r = \{\boldsymbol{L}_1, \ldots, \boldsymbol{L}_r\}$, $0 < r \leq m$, $0 < m \leq 4n'$, and $\boldsymbol{C}$ is a vector confined to a finite region in E. The parameters $\eta_1, \eta_2, \ldots, \eta_m$ are real and positive and will be eventually taken to be large. Let U denote the collection of all subspaces of $R^{4(n+n')}$ such that for any subspace $S \in U$, $\Lambda(I)S = S_r$, where S_r is as defined above.

Similarly, we may write for the renormalised subtracted out *integrand* for its dependence on the external and internal (i.e., integration) variables the expression

$$\begin{aligned} &R(\boldsymbol{L}_1\eta_1 \ldots \eta_s + \ldots + \boldsymbol{L}_r\eta_r \ldots \eta_m + \ldots + \boldsymbol{L}_s\eta_s + \boldsymbol{C}) \\ &\quad = R((\boldsymbol{L}_1\eta_1 \ldots \eta_{r-1}\eta_{r+1} \ldots \eta_s + \ldots + \boldsymbol{L}_r\eta_{r+1} \ldots \eta_s)\eta_r + \ldots + \boldsymbol{L}_s\eta_s + \boldsymbol{C}) \end{aligned}$$

where now $\{\boldsymbol{L}_1, \ldots, \boldsymbol{L}_r\} = S_r \subseteq R^{4(n+n')}$, $0 \leq r \leq s (s \leq 4(n+n'))$ with $\boldsymbol{C}$ confined to a finite region in $R^{4(n+n')}$. We shall say that the parameter η_r is *associated* with the subspace S_r.

In reference to any given subspace $S' \in U$ (that is, for a *given* subspace S' such that $\Lambda(I)S' = S_r$ with S_r *as defined through equation (1)*), the renormalised integrand R may be written, in conformity with the power counting theorem, in our notation as (Zimmermann 1969)

$$R = (1 - T_G) \sum_C Y_G(C), \tag{2}$$

with $Y_G(C)$ defined recursively through

$$Y_{G'}(C) = I_{G'/G_1' \cup \ldots \cup G_n'} \prod_{i=1}^{n} (\delta_{G_i'} - T_{G_i'}) Y_{G_i'}(C), \tag{3}$$

where the sum in equation (2) is over all sets C such that for any $g \in C$ all the $\boldsymbol{V}^{(1)}$ in $g/g_1 \cup \ldots \cup g_c$ are either all orthogonal or all not orthogonal to S'. The subdiagrams $g_1, \ldots, g_c$ are proper, connected and with degree of divergences $d(g_1) \geq 0, \ldots, d(g_c) \geq 0$, and denote the *maximal* elements in C contained in g: $g_i \subset g$, $i = 1, \ldots, c$. In equation (3), $G'(\subseteq G)$ is a proper and connected diagram in C with $G_1', \ldots, G_n'$ as the maximal elements in C contained in the diagram G': $G_i' \subset G'$, $i = 1, \ldots, n$. $T_{G_i'}$ denotes the Taylor operation on the expression $Y_{G_i'}(C)$ with respect to the external variables of the subdiagram G_i' (that is the variables $q_l^{G_i'}$) up to the order $d(G_i')$. The objects $\delta_{G'}$ are defined as follows. If all the $\boldsymbol{V}^{(1)}$ in $G'/G_1' \cup \ldots \cup G_n'$ are not orthogonal to S' and all the $\boldsymbol{V}^{(1)}$ in $G''/G' \cup G''_1 \cup \ldots \cup G''_s$, in the preceding recursive formula for $Y_{G''}$ with $G'' \supset G'$, are, where G'' is in C with $G', G''_1, \ldots, G''_s$ as its maximal elements contained in G'', *then* $\delta_{G'} = 1$, and zero otherwise. $I_{G/G'}$ denotes the unrenormalised Feynman integrand of the diagram G with G' shrunk to a point. Throughout the paper we take the degree of divergence of a diagram to coincide with its dimensionality, and hence $T_g \neq 0$ only if $d(g) \geq 0$.

Consider the following recursive formulae with $G' \subseteq G$ and with the notation $Y_{G'}(C) \equiv Y_{G'}$:

$$
\begin{aligned}
Y_{G'} &= I_{G'/G'_1 \cup \ldots \cup G'_{n'}} \prod_{i=1}^{n'} (\delta_{G'_i} - T_{G'_i}) Y_{G'_i}, \\
Y_{G'_i} &= I_{G'_i/G'_{i1} \cup \ldots \cup G'_{in_i}} \prod_{j=1}^{n_i} (\delta_{G'_{ij}} - T_{G'_{ij}}) Y_{G'_{ij}}, \\
Y_{G'_{ij}} &= I_{G'_{ij}/G'_{ij1} \cup \ldots \cup G'_{ijn_{ij}}} \prod_{k=1}^{n_{ij}} (\delta_{G'_{ijk}} - T_{G'_{ijk}}) Y_{G'_{ijk}},
\end{aligned}
\tag{4}
$$

with the δ defined in reference to a subspace $S' \in U$.

Let c_{ij} be the class of those subdiagrams $\{G'_{ijk}\}$ in equation (4) having $\delta_{G'_{ijk}} = 1$, and let the class of the remaining subdiagrams in $\{G'_{ijk}\}$ with $\delta_{G'_{ijk}} = 0$ be denoted by c'_{ij}. Similarly, we define the classes c_i and c'_i associated with the diagrams in the set $\{G'_{ij}\}$, and classes c and c' associated with the diagrams in the set $\{G'_i\}$.

Lemma 1. Suppose that the class c_{ij} is not empty ($c_{ij} \neq \emptyset$), that is for some G_{ijk}, $\delta_{G_{ijk}} = 1$ by definition; then

$$
\deg_\lambda (\delta_{G'_i} - T_{G'_i}) Y_{G'_i} < d(G'_i) - M(G'_i), \tag{5}
$$

for G'_i in c or in c'. In equation (5) all the momenta internal and/or external in those lines in G'_i with their $\boldsymbol{V}$ not orthogonal to S' have been scaled by λ and the degree is with respect to this parameter. $M(G'_i)$ is the number of independent integration variables characteristic of G'_i and the set C.

The important and non-trivial point to note here is that the equality in (5) is ruled out under the precise stated conditions.

Proof. (i) Case $\delta_{G'_i} = 1$. Then, by definition, $G'_i \in c$ and all the $\boldsymbol{V}^{(1)}$ in $G'_i/G'_{i1} \cup \ldots \cup G'_{in_i}$ are not orthogonal to S'. To carry out the proof, we initially scale only the internal variables of G'_{ijk} with their $\boldsymbol{V}$ not orthogonal to S' by λ, and scale their external variables by a parameter ρ. For $G'_{ijk} \in c_{ij}$, we then have explicitly an expression of the most general form given by

$$
(1 - T_{G'_{ijk}}) Y_{G'_{ijk}} = \sum_{x_{ijk}} (\rho)^{x_{ijk}} F_{x_{ijk}}(\rho, \lambda), \tag{6}
$$

where

$$
\min x_{ijk} \geq 1 + d(G'_{ijk}), \tag{7}
$$

if $d(G'_{ijk}) \geq 0$, and according to the power-counting criterion (Zimmermann 1969),

$$
\max \deg_{\rho,\lambda} F_{x_{ijk}}(\rho, \lambda) \leq d(G'_{ijk}) - M(G'_{ijk}) - x_{ijk}. \tag{8}
$$

The non-negative integer x_{ijk} in (6) is bounded above. Similarly for a $G'_{ijk} \in c'_{ij}$ we have

$$(-T_{G'_{ijk}})Y_{G'_{ijk}} = \sum_{Y_{ijk}} (\rho)^{y_{ijk}} \tilde{F}_{y_{ijk}}(\lambda), \tag{9}$$

where according to the power-counting criterion (Zimmermann 1969),

$$\deg_\lambda \tilde{F}_{y_{ijk}}(\lambda) < -M(G'_{ijk}) \tag{10}$$

for $M(G'_{ijk}) \neq 0$ and

$$\deg_\lambda \tilde{F}_{y_{ijk}}(\lambda) = 0 \tag{11}$$

if $M(G'_{ijk}) = 0$; and where $y_{ijk} \leqslant d(G'_{ijk})$. Note that because the class c_{ij} is assumed not to be empty, it follows that all the $V^{(1)}$ in $G'_{ij}/G'_{ij1} \cup \ldots \cup G'_{ijn_{ij}}$ are necessarily orthogonal to S' according to the subtraction scheme.

In reference to the $(-T_{G'_{ij}})$ operation, the external variables of the G'_{ijk}'s are to be expressed as linear combinations of the internal and external variables of G'_{ij}. Accordingly, by the $(-T_{G'_{ij}})$ operation on $Y_{G'_{ij}}$ we readily obtain symbolically an expression of the form

$$(-T_{G'_{ij}})y_{G'_{ij}} = \sum (\rho)^{x+y+z+w} F_x^{(z)}(\lambda) \tilde{F}_y(\lambda) I^{(w_{ij})}_{G'_{ij}/G'_{ij1} \cup \ldots \cup G'_{ijn_{ij}}}, \tag{12}$$

where symbolically $x \equiv \sum x_{ijk}$, etc, $F^{(a)}(\lambda) = (\partial/\partial\rho)^a F(\rho, \lambda)|_{\rho=0}$, and where we have scaled the external variables in $G'_{ij}/G'_{ij1} \cup \ldots \cup G'_{ijn_{ij}}$ by ρ as well. The summation is over non-negative integers x_{ijk}, y_{ijk}, ... such that

$$x + y + z + w_{ij} \leqslant d(G'_{ij}). \tag{13}$$

Symbolically we also have $F_x^{(z)}(\lambda) \equiv \Pi F^{(z_{ijk})}_{x_{ijk}}$. It is readily seen that $I^{(w_{ij})}_{G'_{ij}/G'_{ij1} \cup \ldots \cup G'_{ijn_{ij}}}$ is independent of λ. We now scale the internal variables in $G'_i/G_{i_1} \cup \ldots \cup G'_{in_i}$ with their V not orthogonal to S' by λ and the external variables by ρ, where $G'_i \in c$. With reference to the $(1 - T_{G'_i})$ operation, we have to express the external variables of the $G'_{ijk} \in c_{ij} \cup c'_{ij}$ as linear combinations of the external and internal variables of G'_i. Accordingly upon the operation of $(1 - T_{G'_i})$ on $Y_{G'_i}$, we obtain, symbolically, an expression

$$\sum (\rho)^{x'+y'+z'+w'} (\lambda)^{x''+y''+z''+w''} F_x^{(z)}(\lambda) \tilde{F}_y(\lambda)$$

$$\times \prod_{(j)} I^{(w_{ij})}_{G'_{ij}/G'_{ij1} \cup \ldots \cup G'_{ijn_{ij}}} (1 - T'_{G'_i}) I_{G'_i/G'_{i1} \cup \ldots \cup G'_{in_i}}, \tag{14}$$

where $T'_{G'_i}$ is of degree $d(G'_i) - x' - y' - z' - w'$, $x' + y' + z' + w' + x'' + y'' + z'' + w'' = x + y + z + w$. Now we scale those external variables of $G'_i/G'_{i_1} \cup \ldots \cup G'_{in_i}$ with their $V^{(2)}$ not orthogonal to S' by λ as well. We then readily check the correctness of the statement in the lemma for $G'_i \in c$.

(ii) Case $\delta_{G'_i} = 0$. Proceeding in the same manner as in the above case, upon writing the external variables of the G'_{ij} as linear combinations of the external and internal variables of G'_i with $G'_i \in c'$, and carrying out the operation in $(-T_{G'_i})Y_{G'_i}$ we obtain from (6)–(13) an expression of the form

$$\sum (\rho)^{x+y+z+w+q} F_x^{(z)}(\lambda) \tilde{F}_y(\lambda) \prod_{(j)} I^{(w_{ij})}_{G'_{ij}/G'_{ij1} \cup \ldots \cup G'_{ijn_{ij}}} I^{(q)}_{G'_{i1} \cup \ldots \cup G'_{in_i}} \tag{15}$$

where $x + y + z + w + q \leqslant d(G'_i)$. Accordingly we have now all those variables, internal and external, in G_i with their $\boldsymbol{V}$ not orthogonal to S', scaled by λ:

$$\max \deg_\lambda \, (-T_{G_i'}) Y_{G_i'} < d(G_i') - M(G_i'), \tag{16}$$

which is the statement of the lemma.

The main point to note in the above lemma is that the upper bound value in (5) cannot be reached, that is an equality does *not* hold for this situation.

This lemma in turn suggests the consideration of the following lemma:

Lemma 2. Let $S \in U$,

$$Y_G(C) = I_{G/G_1 \cup \ldots \cup G_n} \prod_{i=1}^{n} (\delta_{G_i} - T_{G_i}) Y_{G_i}(C), \tag{17}$$

where the $(\delta_{G_i} - T_{G_i}) Y_{G_i}(C)$ are in one of the following forms.

$$\text{(i)} \qquad (1 - T_{G_i}) I_{G_i/G_{i1} \cup \ldots \cup G_{in_i}} \prod_{j=1}^{n_i} (-T_{G_{ij}}) Y_{G_{ij}}(C), \tag{18}$$

with all the δ for all the diagrams in C contained in all the G_{ij} at zero. By definition then all the $\boldsymbol{V}^{(1)}$ in $G_i/G_{i_1} \cup \ldots \cup G_{in_i}$ are not orthogonal to S and all those in $G/G_1 \cup \ldots \cup G_n$ are.

$$\text{(ii)} \qquad (-T_{G_i}) I_{G_i/G_{i1} \cup \ldots \cup G_{in_i}} \prod_{j=1}^{n_i} (-T_{G_{ij}}) Y_{G_{ij}}(C), \tag{19}$$

where all the $\boldsymbol{V}^{(1)}$ in G_i are orthogonal to S, but at least some of the $\boldsymbol{V}^{(2)}$ of G_i are not orthogonal to S (this latter condition on the $\boldsymbol{V}^{(2)}$ may be formally relaxed if $d(G_i) \equiv 0$ by dimensional analysis alone).

Then

$$\deg_\lambda (\delta_{G_i} - T_{G_i}) Y_{G_i} \leqslant d(G_i) - M(G_i), \tag{20}$$

where we have scaled the variables, internal and external, in G_i with their $\boldsymbol{V}$ not orthogonal to S by λ.

Proof. Consider the case (i) with $G_{ij} \equiv \varnothing$ for all $j = 1, 2, \ldots, n$; then it is readily seen by the explicit application of $(1 - T_{G_i})$ on I_{G_i} that

$$\deg_\lambda (1 - T_{G_i}) I_{G_i} \leqslant d(G_i) - M(G_i), \tag{21}$$

where the equality holds if in G_i, with $d(G_i) \geqslant 0$, necessarily some of the $\boldsymbol{V}^{(2)}$ in G_i are not orthogonal to S, by dimensional analysis alone. For $d(G_i) < 0$, the equality in (21) holds trivially.

Consider the case (i) with the $G_{ij} \neq \varnothing$, and suppose, only for concreteness and without loss of generality, that all the $\boldsymbol{V}^{(1)}$ in the G_{ij} are not orthogonal to S. It is easily checked as before and by the explicit Taylor operations, as carried out in lemma 1, that for $d(G_i) \geqslant 0$, the equality in (20) may hold, from dimensional analysis alone, if at least some of the $\boldsymbol{V}^{(2)}$ in each $G_{i1} (\neq \varnothing)$ are not orthogonal to S (this condition on the $\boldsymbol{V}^{(2)}$ may be formally relaxed, from dimensional analysis alone, for those G_{ij} with $d(G_{ij}) \equiv 0$) and if at least some of the $\boldsymbol{V}^{(2)}$ in $G_i/G_{i1} \cup \ldots \cup G_{in_i}$ are not orthogonal to S. For $d(G_i) < 0$ the equality in (20) may hold, from dimensional analysis alone, if at least some of the $\boldsymbol{V}^{(2)}$ in each G_{ij} $(\neq \varnothing)$ are not orthogonal to S (again this latter condition for the $\boldsymbol{V}^{(2)}$ may be relaxed for those G_{ij} with $d(G_{ij}) \equiv 0$ from dimensional analysis alone).

That the equality in (20) for the case (ii) may hold follows by noting that the $(-T_{G_i})$ operation on its preceding expression replaces it by a polynomial of degree $\leq d(G_i)$ in its external variables.

Now we sum over all the sets C in (2) and give, for a given subspace $S \in U$, precise conditions under which an equality corresponding to that in (20) (that is, a maximum dimensionality) for the subtracted-out integrand R may hold from dimensional analysis in *conformity* with the power counting theorem.

Let S be a given subspace in U. Suppose G' is a subdiagram of G such that $I_{G/G'}$ is independent of λ, the parameter associated with S in the expression for R. Here all the momenta, internal and/or external, in those lines in G with their $\boldsymbol{V}$ not orthogonal to S have been scaled by λ. Let $G'_1, \ldots, G'_n$ be the connected components of G'. Let $G'_{i1}, \ldots, G'_{iN_i}$ denote the proper and connected parts of G'_i, $i = 1, 2, \ldots, n$, and $G'_{i0} = \bigcup_{j=1}^{N_i} G'_{ij}$. By definition, with $G'_i \supset G'_{i0}$, G'_i is constructed out of $G'_{i1}, \ldots, G'_{iN_i}$ with the latter connected with one another by singly connecting lines, that is G'_i/G'_{i0} involves no closed loops. Let all the $\boldsymbol{V}$ associated with the external variables in G'_i/G'_{i0} be not orthogonal to S for all $i = 1, 2, \ldots, n$. In general, let $G'_{ij1}, G'_{ij2}, \ldots$ be any proper and connected (if $\neq \varnothing$) subdiagrams contained in a G'_{ij} with $G'_{ij1} \cap G'_{ij2} \neq \varnothing$, pairwise, and $d(G'_{ij1}) \geq 0$, $d(G'_{ij2}) \geq 0, \ldots$, with $j \in [1, 2, \ldots, N_j]$ and $i \in [1, 2, \ldots, n]$ *such that* all the $\boldsymbol{V}$ associated with the internal variables of $G'_{ij1}, G'_{ij2}, \ldots$ are orthogonal to S.

Let

$$L(S) \equiv \sum_{\substack{1 \leq j \leq N_i \\ 1 \leq i \leq n}} L(G'_{ij}/G'_{ij1} \cup G'_{ij2} \cup \ldots), \tag{22}$$

where the sum in (22) is restricted to those terms with all the associated $\boldsymbol{V}$ with the internal variables in $G'_{ij}/G'_{ij1} \cup G'_{ij2} \cup \ldots$ not orthogonal to S. $L(g)$ denotes the number of independent loops in g.

Finally the subspace S and the subdiagram G' are chosen such that the following are true:

(a) Let $\tilde{G}'$ be any other diagram, constructed similarly to G', with corresponding subdiagrams $\tilde{G}'_{ij}, \ldots, G'_{ijk}, \ldots$ as defined for G' with $\tilde{G}'$ formally obtained from G' by adding or deleting a set of lines and vertices to G' with

$$L(S) \equiv \sum{}' L(\tilde{G}'_{ij}/\tilde{G}'_{ij1} \cup \tilde{G}'_{ij2} \cup \ldots),$$

in reference to S, similarly defined as in (22). *Then* G' is such that

$$d(\tilde{G}') \leq d(G'). \tag{23}$$

(b) For any proper and connected diagrams

$$\tilde{G}_B \supseteq \bigcup_{s=1}^{k} G'_{i_s}, \ldots, \tilde{G}_{B'} \supseteq \bigcup_{s'=1}^{k'} G'_{j_{s'}}, \tag{24}$$

(if any), where

$$\{G'_{i_1}, \ldots, G'_{i_k}\}, \ldots, \{G'_{j_1}, \ldots, G'_{j_k}\}, \ldots, \tag{25}$$

are *any* disjoint subsets of the set $\{G'_1, \ldots, G'_n\}$ with

$$\tilde{G}_B \cap \tilde{G}_{B'} = \varnothing \qquad \text{pairwise}, \tag{26}$$

and $I_{(\cup \tilde{G}_B) \cup (\bigcup_{i=1}^{n'} G'_i)/\bigcup_{i=1}^{n} G'_i}$ is independent of λ, where $\bigcup_B \tilde{G}_B$ is the union of the proper and connected diagrams in (24) corresponding to the subsets in (25) and $\bigcup_{i=1}^{n'} G'_i$

corresponds to the subdiagrams *not* appearing in the subsets in (25). *Then* G' is such that

$$\sum_B d(\tilde{G}_B)+\sum_{i=1}^{n}{}' d(G'_i)\leq \sum_{i=1}^{n} d(G'_i)=d(G'), \tag{27}$$

for *any* possible subsets as in (25) with any possible proper and connected diagrams as in (24), (26). Equation (27) is also true for diagrams $\tilde{G}', \ldots$, as given in (a) above with the extreme left-hand side of (27) replaced by corresponding expressions for $\tilde{G}'_B$ defined in reference to $\tilde{G}'$ and the sum $\Sigma'_i\, d(G'_i)$ replaced by a corresponding expression $\Sigma'_i\, d(\tilde{G}'_i)$ with the right-hand side of (27) unchanged.

The above construction is necessary to take all the terms in the expansion in (2) into consideration and to obtain the desired upper bound of R. The condition (23) merely guarantees the fact that the degree of R, with respect to λ, cannot be increased further beyond the right-hand side value of equation (28) (see below) by a rearrangement of subdiagrams. The condition (27) guarantees the fact that we cannot find Taylor operations corresponding to the subdiagrams $\tilde{G}_B, \ldots, \tilde{G}_{B'}, \ldots$ in (24) and (26) which may further increase $\deg_\lambda R$ beyond the upper bound value given in equation (28). The latter follows from the following. Let

$$Y_{\tilde{G}_B}=I_{\tilde{G}_B/G'_{ij}\cup G'_{kl}\cup G'_{mn}\cup\ldots}(1-T_{G'_{ij}})Y_{G'_{ij}}(1-T_{G'_{kl}})Y_{G'_{kl}}(1-T_{G'_{mn}})Y_{G'_{mn}}\ldots,$$

be an expression corresponding to a subset in (25) such that

$$\deg_\lambda(1-T_{G'_{ij}})Y_{G'_{ij}}\leq d(G'_{ij})-4L_{ij}(S),$$

where $L_{ij}(S)\neq 0$, is the number of independent loops in $G'_{ij}/G'_{ij1}\cup G'_{ij2}\cup\ldots$, for example. Then we have explicitly for $\tilde{G}_B\supset \bigcup_s G'_{i_s}$, with $d(\tilde{G}_B)\geq 0$,

$$\deg_\lambda(-T_{\tilde{G}_B})Y_{\tilde{G}_B}<d(\tilde{G}_B)-4\sum_{[(i,j),(k,l),\ldots]}{}' L_{ij}(S).$$

Summing over all such $\tilde{G}_B$ in (24) leads to the condition stated in equation (27). On the other hand (27) and (23) are equivalent if, for example, the diagram $(\bigcup_B \tilde{G}_B)\cup(\bigcup_{i=1}^{n'} G'_i)$ coincides with $\tilde{G}'$. Condition (27) also guarantees the fact that we cannot find a subspace $\tilde{S}\in U$, as S given above, which may increase further the expression $\deg_{\tilde{\lambda}} R+4L(\tilde{S})$, beyond the value $d(G')$, where $\tilde{\lambda}$ is the parameter associated with $\tilde{S}$ and $L(\tilde{S})$ is defined similarly to $L(S)$.

Upon summing over all the sets C in equation (2), in reference to a subspace $S\in U$, the above analysis together with the two lemmas establish completely the following theorem. This theorem is basic for the investigation of the asymptotic polynomial and logarithmic behaviour of renormalised Feynman amplitudes, in conformity with the power-counting theorem.

Theorem.

$$\deg_\lambda R\leq d(G')-4L(S) \tag{28}$$

with $L(S)$ as defined in equation (22), and where the equality in equation (28) may hold, from dimensional analysis alone, if one of the following is true for each $G'_{ij}\subset G_i$, with $G'=\bigcup_{i=1}^{n} G'_i$, in each of the following cases:

2.1. Case I

All the $V^{(1)}$ in a $G'_{ij}/G'_{ij1}\cup G'_{ij2}\cup\ldots$ are not orthogonal to S.

(i) $d(G'_{ij})<0$, $G'_{ij1}, G'_{ij2}, \ldots=\varnothing$.

(ii) $d(G'_{ij}) \geqslant 0$, $G'_{ij1}, G'_{ij2}, \ldots = \varnothing$, and some of the $\boldsymbol{V}^{(2)}$ in G'_{ij} are not orthogonal to S.

(iii) $d(G'_{ij}) \geqslant 0$ or $d(G'_{ij}) < 0$, $G'_{ij1}, G'_{ij2}, \ldots \neq \varnothing$ and at least some of the $\boldsymbol{V}^{(2)}$ in each $G'_{ij1}, G'_{ij2}, \ldots$ are not orthogonal to S (the latter condition on the $\boldsymbol{V}^{(2)}$ may be formally relaxed from dimensional analysis alone for those G'_{ijk} with $d(G'_{ijk}) \equiv 0$).

2.2. Case II

All the $\boldsymbol{V}^{(1)}$ in a G'_{ij} are orthogonal to S.

(iv) $d(G'_{ij}) \geqslant 0$ and some of the $\boldsymbol{V}^{(2)}$ in G'_{ij} are not orthogonal to S.

In the case when the $\boldsymbol{V}^{(1)}$ in G' are orthogonal to S, then the above criteria (i)–(iv) collapse to the last one (iv) for all G'_{ij} with $L(S) = 0$.

3. Conclusion

We have carried out a rigorous dimensional analysis of Bogoliubov subtracted-out Feynman integrands. The criteria under which, as stated in (i)–(iv) in the theorem, there is an equality in (28) (that is, yielding a maximum dimensionality for R in conformity with the power-counting theorem) give us the sufficient conditions to determine the asymptotic polynomial and logarithmic behaviour of A when $\eta_1, \ldots, \eta_r, \ldots, \eta_m$, as appearing in the argument of A in equation (1), become independently large. In reference to a subspace S' in U, define the degree of the bound of R (that is, the so-called asymptotic coefficient) by $\alpha(S')$ as the power of, say, a parameter λ. Here all those internal and/or external variables in G (the graph in question with which the subtracted-out amplitude A is associated) with their $\boldsymbol{V}$ not orthogonal to S' have been scaled by λ. Let $G' \subseteq G$ be the subdiagram consisting of all such lines and the vertices joining them. If G' associated with the subspace S' (that is, all its $\boldsymbol{V}$ are not orthogonal to S') is *such that* for *any* $S'' \in U$ (G'' is a subdiagram associated with it, similarly defined), $d(G'') \leqslant d(G')$, *then* $\alpha_I(S_r) = d(G')$ with $\dim S' - \dim S_r = 4L(S') = 4L(G')$. Here $\dim S'$ denotes the dimension of the space S'. The parameter $\alpha_I(S_r)$ denotes the so-called asymptotic coefficient of the integral of the integrand R, that is, that of the subtracted-out amplitude A, and $\dim S'$ denotes the dimension of the space S'. This result follows from the well known relation (Weinberg 1960)

$$\alpha_I(S_r) = \max_{\Lambda(I)S = S_r} [\alpha(S) + \dim S - \dim S_r],$$

from the definition of S', and from the theorem given above with a contribution to R (the sum over the sets C in equation (2)) coming in conformity with the power-counting theorem, for example, from an expression as in equation (17) in lemma 2 with $G_{ij} = \varnothing$ for all j and $L(S') = \Sigma_i L(G_i)$, for $L(S') \neq 0$, and $G_i = \varnothing$ for $L(S') = 0$, for all i. The space S' is then called a 'maximising subspace' for the bound of A relative to the parameter η_r, with $\alpha_I(S_r)$ as the power of η_r. Thus we see that the polynomial or power behaviour of A is straightforwardly obtained. The class of all such subspaces as S', as just defined, however, does *not* exhaust all the maximising subspaces for the bound of A and consists only of a subclass of the larger class of all the maximising subspaces. Our theorem gives us the precise information to construct the larger class of all maximising subspaces (Manoukian 1978). This theorem shows also that the smaller class just discussed above, to which the space S' given above belongs, is not empty and is indeed a subclass of the larger class. This latter analysis is necessary for the determination of the logarithmic behaviour of A as well. Accordingly, a study of the asymptotic behaviour of A without actually carrying out the subtractions of renormalisation is necessarily incomplete. Such an analysis is given in detail in Manoukian (1978) and gives both the polynomial (as discussed briefly above) and the logarithmic behaviour when all or, more generally, some of the external momentum components of the graph in question become large in Euclidean space non-exceptionally.

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